HYPERBOLIC DIFFUSION EQUATION IN A RELAXATIONAL AND DISPERSED MEDIUM

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Using an operator representation of the effective diffusion coefficient we obtain a hyperbolic diffusion equation in a relaxational medium.

We assume that in the system there occurs a chemical reaction or a certain relaxational process described by the phenomenological equation

$$\hat{\mathrm{D}}\xi = LA, \tag{1}$$

where $\hat{\mathrm{D}}$ = d/dt is the operator of differentiation with respect to time and L is a phenomenological coefficient. The affinity of process A in equilibrium vanishes. For a pair of thermodynamic parameters y and x we can introduce an effective thermodynamic derivative [1]

$$\left(\frac{\partial \hat{y}}{\partial x}\right)_z = \left(\frac{\partial y}{\partial x}\right)_{z\xi} + \frac{\left(\frac{\partial y}{\partial x}\right)_{zA} - \left(\frac{\partial y}{\partial x}\right)_{z\xi}}{1 + \tau_{zx}\hat{\mathrm{D}}} . \tag{2}$$

The relaxation time τ_{zx} can be expressed by the usual method [2] in terms of the affinity A:

$$\tau_{zx} = -\left[L\left(\frac{\partial A}{\partial \xi}\right)_{zx}\right]^{-1} . \tag{3}$$

The index $\xi(\infty)$ corresponds to the instantaneous ("frozen") states; A(0) to the equilibrium states.

In [1], using Eq. (2) there were introduced operator representations of the adiabatic exponent, specific heat, and velocity of sound; in [3], by a somewhat different method, operator representations of the thermal coefficients were introduced.

Using (2) for the Gibbs function (y = G) and the number of particles of definite type (x = N) for Z = P, T, we obtain the operator representation of the chemical potential $\mu = (\partial G/\partial N)_{P,T}$:

$$\hat{\mu} = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + \tau_{P,T}\hat{\mathrm{D}}} Z = P, T, . \tag{4}$$

We next consider single-component diffusion in an isothermal medium (this assumption is not fundamental, and the results obtained below can be extended to the case of multicomponent diffusion for the presence of gradients of temperature and of pressure).

We apply the operator ∇ to both sides of (4), assuming that the chemical potential is a function of the concentration of the diffusing component C:

$$\left(\frac{\partial \hat{\mu}}{\partial C}\right)_{P,T} \cdot \nabla C = \left[\left(\frac{\partial \mu_\infty}{\partial C}\right)_{P,T} + \frac{\left(\frac{\partial \mu_0}{\partial C}\right)_{P,T} - \left(\frac{\partial \mu_\infty}{\partial C}\right)_{P,T}}{1 + \tau_{P,T}\hat{\mathrm{D}}}\right] \cdot \nabla C. \tag{5}$$

Writing Fick's law in the operator form

$$\hat{\mathrm{J}} = -L\nabla\hat{\mu} \tag{6}$$

and taking (5) into account, we obtain

$$\hat{\mathrm{J}} = -\hat{L}\left(\frac{\partial \hat{\mu}}{\partial C}\right)_{P,T} \cdot \Delta C = -\rho\hat{D}\nabla C, \tag{7}$$

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where in accordance with [4] the diffusion coefficient is determined from $\hat{D} = L^{-1}\rho \left(\frac{\partial \mu}{\partial C}\right)_{P,T}$; ρ is the density of the substance.

From (5) and (7), the operator form of the diffusion coefficient

$$\hat{D} = D_\infty + \frac{D_0 - D_\infty}{1 + \tau\hat{\mathrm{D}}} \tag{8}$$

follows. We consider the mass balance for a volume V, in which the diffusion flow is determined by Eq. (7), and the sources σ act:

$$\int_v \rho \frac{\partial C}{\partial t} dV = -\int_\Omega \hat{J} d\Omega + \int_v \sigma dV.$$

Substituting Eqs. (7) and (8) into this expression and assuming ρ = const, we obtain the hyperbolic diffusion equation

$$\frac{\partial C}{\partial t} + \tau_T \frac{\partial^2 C}{\partial t^2} = C_\infty \tau_T \frac{\partial}{\partial t} \nabla^2 C + D_0 \nabla^2 C + \left(1 + \tau_T \frac{\partial}{\partial t}\right) \sigma. \tag{9}$$

The formalism considered above, in principle, can also be used for describing the diffusion of particles of a dispersed medium. In this case, following the general ideas of [5], we can add to the thermodynamic potential of the system the nonthermodynamic contribution G′, depending on the dispersed particles in the system:

$$G_d = G + G'. \tag{10}$$

The calculation of this contribution or the direct calculation of the chemical potential $\mu_d = \frac{\partial G_d}{\partial N} = \frac{\partial G'}{\partial N}$ of the dispersed particles is a very complicated problem. The chemical potential was calculated in [6] for moderately concentrated dispersed systems. For the approach considered here, however, it is not necessary to have an explicit expression for the thermodynamic or chemical potential.

Thus, applying the procedure considered above for the derivation of the hyperbolic diffusion equation to a dispersed system with thermodynamic potential (10) or chemical potential μ_d, we again arrive at Eq. (10) and the operator representation of the diffusion coefficient of the dispersed particles (8). Since, within the framework of the thermodynamic approach, D_∞ and D_0 represent phenomenological coefficients, we can consider them as empirically determinable quantities.

We illustrate the applicability of such an approach for the example of diffusion in an inhomogeneous fluidized bed. An approximate stochastic model of the process was proposed by Pakhaluev [7].*

Assuming ξ in (1) to be a random displacement of the aggregates of particles and representing A in the form

$$A = \left(\frac{\partial A}{\partial \xi}\right)_X \cdot \Delta\xi + \left(\frac{\partial A}{\partial X}\right)_\xi \cdot \Delta X,$$

after repeated differentiation of (1) with respect to time we arrive at the Langevin equation, based on which the theory in [7] was constructed:

$$\hat{\mathrm{D}}_\xi^2 + \frac{1}{\tau} \hat{\mathrm{D}}_\xi = F(t), \tag{11}$$

where τ is a certain characteristic time, and $F(t) = L\left(\frac{\partial A}{\partial X}\right)_\xi \hat{\mathrm{D}} X$ is the random action on the aggregates of particles from the direction of the gas bubbles. Analysis of the spectral form of the solution of Eq. (11) [7] shows that the diffusion coefficient depends on the natural frequency ω_0 of the fluidized system. By replacing the operator $\hat{\mathrm{D}}$ in (8) by the eigenvalue $i\omega_0$, we obtain the spectral representation of the diffusion coefficient

$$D(\omega) = D_\infty + \frac{D_0 - D_\infty}{1 + i\omega_0\tau}. \tag{12}$$

Here D_0 is the diffusion coefficient for $\omega_0 = 0$. Separating (12) into real and imaginary parts we obtain

$$D = D_\infty + \frac{D_0 - D_\infty}{1 + \omega_0^2\tau^2} + i(D_0 - D_\infty)\frac{\omega_0\tau}{1 + \omega_0^2\tau^2}. \tag{13}$$

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*For greater detail see V. M. Pakhaluev, "Investigation of the process of heat transport in retarded fluidized beds," Candidate's Dissertation, Sverdlovsk (1969).

The diffusion coefficient is determined as Re(D). If we neglect diffusion transport for retarded states $D_\infty \sim 0$, then, taking into account the expression for D_0 obtained in [7], the coefficient has the form

$$D_d = \frac{D_0}{1+\omega_0^2\tau^2} = K(W - W_{\mathrm{cr}})\frac{\tau^2}{1+\omega_0^2\tau^2}, \tag{14}$$

where K is a certain parameter, and $W - W_{\mathrm{cr}}$ is the excess velocity with respect to the critical velocity. This also is a basic result of [7]. It was confirmed by experiment from the definition of the statistical characteristics of fluidized systems.

The representation of the operator $[1+\tau\hat{D}]^{-1}$, derived in [1] and used in the present study, in the form of a series (the idea for such a representation came from Yu. A. Buevich) enables us (in an approximation that is linear with respect to $\tau\hat{D}$) to obtain a somewhat different form of Eq. (9):

$$\frac{\partial C}{\partial t} + \tau\left(1+\frac{D_\infty}{D_0}\right)\frac{\partial^2 C}{\partial t^2} = D_0\nabla^2 C. \tag{15}$$

The membership of the last equation to a definite type (hyperbolic, parabolic, or elliptic) depends on the relation between the transport coefficients D_∞ and D_0. A similar conclusion can also be obtained for the heat equation [8].

Equation (15) can be used for characterizing the transport of the material from a binary fluidized layer as applied to the conditions for carrying out the experiment described in [9]. The dynamics for transporting (carrying) small particles from the separator is determined by the relation of the flow for the entrainment of small particles from the surface of the fluidized bed (q_{en}) and the admission of the material from the mixing chamber (q_0). A mathematical model of the process includes Eq. (15), taking into account the finiteness of the velocity of motion of the fine particles in the apparatus and the boundary conditions determining the flows q_0 and q_{en}:

$$-D\left.\frac{\partial C}{\partial X}\right|_{X=0} = q_0, \tag{16}$$

$$-D\left.\frac{\partial C}{\partial X}\right|_{x=h} = q_{\mathrm{en}}. \tag{17}$$

In the first approximation, taking into account the considerably larger volume of the mixing chamber in comparison with the separator the flow q_0 can be assumed constant ($q_0 \approx \mathrm{const}$). The quantity q_{en} is determined as the rate of the first-order chemical reaction. For convenience in subsequent calculations we determine q_{en} according to the value of the mean concentration of fine particles in the separator $\bar{C} = \frac{1}{h}\int_0^h C\,dX$:

$$q_{\mathrm{en}} = \beta\bar{C}. \tag{18}$$

Integrating (15) with respect to X in the limits from 0 to h and converting to fine-particle concentrations $\bar{C}$, averaged over the height of the separator, with account of (16)-(18) we have an equation for determining $\bar{C}(t)$

$$h\left[1+B\frac{\partial}{\partial t}\right]\frac{\partial\bar{C}}{\partial t} = -\beta\bar{C} + q_0, \tag{19}$$

where h is the height of the layer in the separator, and $B \equiv \tau(1 - D_\infty/D_0)$.

Taking into account the absence of "fine particles" in the separator at the initial time ($\bar{C}|_{t=0} = 0$), we integrate (19) by the perturbation method. In an approximation that is linear with respect to B, we obtain an expression that determines the rate of entrainment of the fine particles from the separator:

$$q_{\mathrm{en}} = q_0\left\{1 - e^{-\frac{\beta t}{h}}\left(1 - \frac{\beta^2 Bt}{h^2}\right)\right\}. \tag{20}$$

Figure 1 shows experimental data on the entrainment rate for various values of the regime parameters of the process. As can be seen from Fig. 1, the form of the curves corresponds to relations describable by Eq. (20) for various values of B.

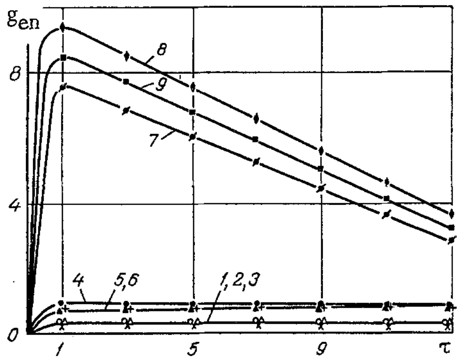

Fig. 1. Rate of entrainment as a function of time for different velocities in the separator (W) and different initial heights of the bed (h_{bed}): 1, 4, 7) h_{bed} = 137; 3, 6, 9) 166 mm; 2, 5, 8) 200 mm; 1, 2, 3) W = 1.1; 4, 5, 6) 1.4; 7, 8, 9) 1.8 m/sec, τ, min; g_{en}, g/sec.

NOTATION

F_{slot}, return-flow slot area; F, cross-sectional area of the bed in the vertical plane; G*, instantaneous number of "fine particles" in the apparatus.

LITERATURE CITED

1. G. P. Yasnikov and V. S. Belousov, "Effective thermodynamic functions of a gas with solid particles," Inzh.-Fiz. Zh., 34, No. 6, 1085-1089 (1978).
2. S. R. DeGroot and P. Mazur, Nonequilibrium Thermodynamics, American Elsevier (1962).
3. John Lamb, "Thermal relaxation in liquids," in: Physical Acoustics: Principles and Methods, W. P. Mason (editor), Vol. 2A, Academic (1965).
4. A. Katchalsky and P. F. Curran, Nonequilibrium Thermodynamics in Biophysics, Cambridge, Mass. (1965).
5. S. A. Boguslavskii, Selected Works in Physics [in Russian], Fizmatgiz, Moscow (1961).
6. Yu. A. Buevich and A. A. Ryvkin, "The structuring of moderately concentrated, finely dispersed systems. Equilibrium properties," Kolloidn. Zh., 41, No. 4, 632-640 (1979).
7. V. M. Pakhaluev, "Stochastic model of a transport process in an inhomogeneous fluidized system," in: Hydrodynamics and Heat Exchange [in Russian], Akad. Nauk SSSR, UNTs, Sverdlovsk (1974), pp. 52-56.
8. Yu. A. Buevich and Yu. A. Korneev, "Dispersion of thermal waves in a granular material," Inzh.-Fiz. Zh., 31, No. 1, 21-28 (1976).
9. A. P. Baskakov, Z. N. Kutyavin, L. G. Gal'perin, and N. V. Ezhova, "Investigation of diffusion of a fine granular material in a binary fluidized bed," in: Physics of Aerodispersed Systems [in Russian], No. 14, Vishcha Shkola, Kiev–Odessa (1976), pp. 122-123.
10. Max Leva, Fluidization, McGraw-Hill (1959).